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pp-waves in 11-dimensions with extra supersymmetry

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Abstract: The Killing spinor equations for pp-wave solutions of eleven dimensional supergravity are analysed and it is shown that there are solutions that preserve 18, 20, 22 and 24 supersymmetries, in addition to the generic solution preserving 16 supersymmetries and the Kowalski-Glikman solution preserving 32 supersymmetries.

Keywords: M-Theory, Penrose limit and pp-wave background.

 http://jhep.sissa.it/archive/papers/jhep062002013/jhep062002013.pdf

Eleven-dimensional supergravity has pp-wave solutions [1]

$$
\begin{aligned}
ds^2 &= 2dx^+dx^- + H(x^i,x^-)(dx^-)^2 + \sum_{i=1}^{9}(dx^i)^2 \\
F_4 &= dx^- \wedge \xi(x^i,x^-)
\end{aligned}
\tag{1}
$$

where $H(x^i,x^-)$ obeys

$$
\triangle H = -\frac{1}{4}\|\xi\|^2 \,. \tag{2}
$$

Here $\triangle$ is the laplacian in the transverse euclidean space $\mathbb{E}^9$ with coordinates x^i and $\xi(x^i,x^-)$ is (for each x^-) a closed and coclosed 3-form in $\mathbb{E}^9$. This solution has at least 16 Killing spinors.

An interesting subclass of these metrics are those for which

$$
H(x^i,x^-) = \sum_{i,j} A_{ij}x^ix^j \tag{3}
$$

where $A_{ij} = A_{ji}$ is a constant symmetric matrix [2]. In particular, this class contains a maximally supersymmetric solution with 32 Killing spinors [3, 4]

$$
\begin{aligned}
A_{ij} &= \begin{cases} -\frac{1}{9}\mu^2\delta_{ij} & i,j=1,2,3 \\ -\frac{1}{36}\mu^2\delta_{ij} & i,j=4,5,\ldots,9 \end{cases} \\
\xi &= \mu dx^1 \wedge dx^2 \wedge dx^3 \,,
\end{aligned}
\tag{4}
$$

where μ is a parameter which can be set to any nonzero value by a change of coordinates.

In [5], a similar maximally supersymmetric solution of IIB supergravity was found, and in [6, 7] it was shown that both of these solutions arise as Penrose limits [8] of maximally supersymmetric $AdS \times$ Sphere solutions. The IIB string theory in this background can be exactly solved [9] and is dual to a certain subsector of $N = 4$ supersymmetric Yang-Mills theory [7]. Subsequent work developing these ideas includes [11]-[27].

Given that pp-waves generically preserve at least half of the supersymmetries, and for special cases preserve all of the supersymmetries, it is natural to ask whether there are similar solutions preserving fractions ν of the supersymmetry with $1/2 < \nu < 1$. In [28], it was argued that configurations preserving such fractions of supersymmetry could arise in M-theory. In [21] it was shown that such configurations do indeed arise as IIB pp-waves, and a pp-wave of M-theory preserving 3/4 supersymmetry was presented in [25]. The purpose here is to investigate pp-wave solutions of 11-dimensional supergravity in more detail, and to show that the fractions 9/16, 5/8, 11/16 can also arise in addition to 3/4, 1/2 and 1.

Our ansatz is

$$
\begin{aligned}
ds^2 &= 2dx^+dx^- + \sum_{i,j} A_{ij}x^ix^j(dx^-)^2 + \sum_i dx^idx^i \\
F &= dx^- \wedge \xi \,,
\end{aligned}
\tag{5}
$$

where ξ is a 3-form on $\mathbb{E}^9$ with constant coefficients. This is a supersymmetric solution of eleven-dimensional supergravity, provided that

$$\operatorname{tr} A = -\frac{1}{2}\|\xi\|^2 = -\frac{1}{12}\xi_{ijk}\xi^{ijk}\,, \tag{6}$$

with $i, j, k = 1, \ldots, 9$. We seek the conditions for such solutions to admit Killing spinors, following the analysis of [2] and [5, 21]. We will see that this occurs for specific choices of ξ, A.

The Killing spinors ε satisfy the equation

$$\nabla_M \varepsilon = \Omega_M \varepsilon\,, \tag{7}$$

where ∇ is the spin connection and

$$\Omega_M = \frac{1}{288}\left(\Gamma_M{}^{PQRS} - 8\delta_M^P \Gamma^{QRS}\right) F_{PQRS}\,, \tag{8}$$

In the frame

$$\begin{aligned} e^+ &= dx^+ + \frac{1}{2}\sum_{i,j} A_{ij} x^i x^j dx^- \\ e^- &= dx^- \\ e^i &= dx^i \end{aligned} \tag{9}$$

the only nonvanishing components of the spin connection are

$$\omega^{+i} = \sum_j A_{ij} x^j dx^-\,. \tag{10}$$

We also have

$$\begin{aligned} \Omega_+ &= 0 \\ \Omega_- &= -\frac{1}{12}\Theta\left(\Gamma_+\Gamma_- + 1\right) \\ \Omega_i &= \frac{1}{24}(3\Theta\Gamma_i + \Gamma_i\Theta)\Gamma_+ \end{aligned} \tag{11}$$

where the indices on the left-hand-side are co-ordinate indices, while the indices on the gamma-matrices here, and for the rest of the paper, are frame indices, and

$$\Theta = \frac{1}{6}\xi_{ijk}\Gamma^{ijk}\,. \tag{12}$$

For any such solution, it is simple to see that there are always 16 “standard” Killing spinors satisfying

$$\Gamma_+\varepsilon = 0\,. \tag{13}$$

Explicitly they are given by

$$\varepsilon = \exp\left(-\frac{1}{4}x^-\Theta\right)\psi\,, \tag{14}$$

for some constant spinor ψ such that $\Gamma_+\psi = 0$.

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Next we look for "extra" Killing spinors with $\Gamma_+\varepsilon \neq 0$. It was shown in [2] that *any* Killing spinor is of the form

$$\varepsilon = \left(1 + \sum_i x^i \Omega_i\right) \chi\,, \tag{15}$$

where the spinor χ only depends on x^-. The dependence on x^- is determined by

$$\partial_- \varepsilon = -\frac{1}{2} \sum_{i,j} A_{ij} x^j \Gamma_+ \Gamma_i \varepsilon - \frac{1}{12} \left(\Gamma_+\Gamma_- + 1\right) \Theta \varepsilon\,, \tag{16}$$

which, using (15), gives

$$\begin{aligned} \frac{d}{dx^-}\chi = & -\frac{1}{12}\Theta\left(1 + \Gamma_+\Gamma_-\right)\chi + \\ & + \sum_i x^i \left(-\frac{1}{2}\sum_j A_{ij}\Gamma_+\Gamma_j + \frac{1}{12}\Omega_i\Theta - \frac{1}{4}\Theta\Omega_i\right)\chi\,. \end{aligned} \tag{17}$$

As χ is independent of x^i, this can be decomposed into a piece independent of x^i, and a piece that is linear in x^i. The piece independent of x^i is

$$\frac{d}{dx^-}\chi = -\frac{1}{12}\left(\Gamma_+\Gamma_- + 1\right)\Theta\chi \tag{18}$$

which determines the x^- dependence of χ. The part linear in x^i gives

$$\left(-144\sum_j A_{ij}\Gamma_j + 9\Theta^2\Gamma_i + 6\Theta\Gamma_i\Theta + \Gamma_i\Theta^2\right)\Gamma_+\chi = 0\,. \tag{19}$$

and we now proceed to analyse this.

We choose a representation of the 32×32 Dirac matrices Γ_M in which

$$\Gamma_i = \gamma_i \otimes \sigma_3\,, \qquad \Gamma_\pm = 1 \otimes \sigma_\pm \tag{20}$$

where $i, j = 1, \ldots, 9$, γ_i are 16×16 gamma matrices for SO(9), $(\sigma_1, \sigma_2, \sigma_3)$ are 2×2 Pauli matrices, with $\sigma_\pm = \frac{1}{\sqrt{2}}(\sigma_1 \pm i\sigma_2)$. A 32-component spinor χ then decomposes into two SO(9) spinors, $\chi_\pm$:

$$\chi = (\chi_+, \chi_-)\,, \qquad \Gamma_-\chi = \sqrt{2}(0, \chi_+)\,, \qquad \Gamma_+\chi = \sqrt{2}(\chi_-, 0) \tag{21}$$

Then

$$\Theta = \theta \otimes \sigma_3\,, \qquad \theta = \frac{1}{6}\xi_{ijk}\gamma^{ijk} \tag{22}$$

and $\theta_{\alpha\beta}$ is an antisymmetric 16×16 matrix, where $\alpha, \beta = 1, \ldots, 16$ are SO(9) spinor indices. Equation (18) implies

$$\chi_+ = \exp\left(-\frac{1}{4}x^-\theta\right)\psi_+\,, \qquad \chi_- = \exp\left(-\frac{1}{12}x^-\theta\right)\psi_-\,, \tag{23}$$

for constant 16-component spinors $\psi_\pm$. Equation (19) imposes no further conditions on χ_+.

A Killing spinor in 11 dimensions will only give rise to a Killing spinor in the theory obtained by dimensional reduction in a direction generated by a Killing vector if the Lie derivative in the Killing direction of that spinor vanishes. The spinors $\psi_\pm$ that are anihilated by θ give Killing spinors that are independent of x^- and hence survive under dimensional reduction in the x^- direction. The standard Killing spinors, parametrised by ψ_+, are independent of x^i while the x^i dependence of any extra Killing spinors, parametrised by ψ_-, are encoded in (15). If the matrix A is such that ∂_i is a Killing vector then the Killing spinors independent of x^i will survive under dimensional reduction on this Killing vector.

Next, we need to specify our ansatz for ξ and hence Θ. Any anti-symmetric matrix $L_{\alpha\beta}$ can be written in terms of a 2-form L_{ij} and a 3-form L_{ijk} as

$$L_{\alpha\beta} = \frac{1}{2} L_{ij} (\gamma^{ij})_{\alpha\beta} + \frac{1}{6} L_{ijk} (\gamma^{ijk})_{\alpha\beta} \,. \tag{24}$$

This gives a decomposition of the Lie algebra of SO(16) (the 16×16 antisymmetric matrices $L_{\alpha\beta}$) into the maximal Spin(9) subalgebra (the 9×9 antisymmetric matrices L_{ij}), and its complement (specified by the 3-forms L_{ijk}). Now SO(16) has rank 8 while Spin(9) has rank 4, so any Cartan subalgebra of SO(16) is generated, for some $n \le 4$, by n commuting generators of Spin(9) corresponding to n 2-forms, and $8-n$ commuting elements from the complement of Spin(9), corresponding to $8-n$ 3-forms. Only the cases $n=4$ and $n=1$ occur. A convenient choice for the Cartan subalgebra with $n=4$ is the commuting set of four generators

$$(\gamma^{12})_{\alpha\beta}\,, \qquad (\gamma^{34})_{\alpha\beta}\,, \qquad (\gamma^{56})_{\alpha\beta}\,, \qquad (\gamma^{78})_{\alpha\beta} \tag{25}$$

of Spin(9), together with

$$(\gamma^{129})_{\alpha\beta}\,, \qquad (\gamma^{349})_{\alpha\beta}\,, \qquad (\gamma^{569})_{\alpha\beta}\,, \qquad (\gamma^{789})_{\alpha\beta} \tag{26}$$

A convenient basis with $n=1$ consists of the Spin(9) generator $(\gamma^{89})_{\alpha\beta}$ together with the seven 3-forms

$$\begin{array}{llll} (\gamma_{123})_{\alpha\beta}\,, & (\gamma_{145})_{\alpha\beta}\,, & (\gamma_{167})_{\alpha\beta}\,, & (\gamma_{246})_{\alpha\beta} \\ (\gamma_{257})_{\alpha\beta}\,, & (\gamma_{347})_{\alpha\beta}\,, & (\gamma_{356})_{\alpha\beta} & \end{array} \tag{27}$$

A basis in spin-space can be chosen to bring any given anti-symmetric $\theta_{\alpha\beta}$ to skew-diagonal form with skew eigenvalues $\lambda_1, \dots, \lambda_8$, $\theta = \epsilon \otimes \Lambda$, where $\epsilon = i\sigma_2$ and Λ is the 8×8 diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_8)$. Then the eigenvalues of the symmetric matrix θ^2 are $-\lambda_I^2$, $I = 1, \dots, 8$, each with degeneracy 2. The set of such skew-diagonal matrices generate an 8-dimensional Cartan subalgebra of SO(16), and so can be decomposed into n 2-forms and $8-n$ 3-forms where either $n=1$ or $n=4$, and the set of $\theta_{\alpha\beta}$ are stratified into distinct orbits with $n=1$ or $n=4$.

Now the flux ξ_{ijk} determines a 16×16 anti-symmetric matrix $\theta_{\alpha\beta}$ from (22). However, it is not an arbitrary antisymmetric matrix, but one for which the 2-form part vanishes, i.e. one satisfying the constraint $(\gamma^{ij})^{\alpha\beta}\theta_{\alpha\beta} = 0$. If it occurs in the $n=4$ orbit, it can be written in terms of four linearly independent 3-forms using (24), and using SO(9) transformations, these can be arranged to be precisely the generators in (26). That is, one can choose bases for spin-space and the tangent space such that the generators (25), (26) are skew-diagonal and $\theta_{\alpha\beta}$ is a linear combination of the 3-form generators (26) alone, so that there are constants m_1, m_2, m_3, m_4 such that

$$\theta_{\alpha\beta} = m_1(\gamma_{129})^{ab} + m_2(\gamma_{349})^{ab} + m_3(\gamma_{569})^{ab} + m_4(\gamma_{789})^{ab} \,. \tag{28}$$

The skew eigenvalues $\lambda_1, \dots, \lambda_8$ of θ are then, in a convenient basis, given by

$$\begin{aligned}
\lambda_1 &= -m_1 - m_2 + m_3 - m_4 \\
\lambda_2 &= m_1 + m_2 - m_3 - m_4 \\
\lambda_3 &= m_1 + m_2 + m_3 - m_4 \\
\lambda_4 &= -m_1 - m_2 - m_3 - m_4 \\
\lambda_5 &= -m_1 + m_2 + m_3 + m_4 \\
\lambda_6 &= m_1 - m_2 - m_3 + m_4 \\
\lambda_7 &= m_1 - m_2 + m_3 + m_4 \\
\lambda_8 &= -m_1 + m_2 - m_3 + m_4
\end{aligned} \tag{29}$$

Similarly, if $\theta_{\alpha\beta}$ lies in the $n = 1$ orbit, one can choose bases such that

$$\begin{aligned}
\theta_{\alpha\beta} = {} & n_1(\gamma_{123})_{\alpha\beta} + n_2(\gamma_{145})_{\alpha\beta} + n_3(\gamma_{167})_{\alpha\beta} + n_4(\gamma_{246})_{\alpha\beta} + \\
& + n_5(\gamma_{257})_{\alpha\beta} + n_6(\gamma_{347})_{\alpha\beta} + n_7(\gamma_{356})_{\alpha\beta}
\end{aligned} \tag{30}$$

for some constants $n_1, n_2, \dots, n_7$. The skew eigenvalues $\lambda_1, \dots, \lambda_8$ of θ are then, in a convenient basis, given by

$$\begin{aligned}
\lambda_1 &= -n_1 - n_2 - n_3 - n_4 + n_5 + n_6 + n_7 \\
\lambda_2 &= -n_1 + n_2 + n_3 + n_4 - n_5 + n_6 + n_7 \\
\lambda_3 &= n_1 + n_2 - n_3 - n_4 - n_5 - n_6 + n_7 \\
\lambda_4 &= n_1 - n_2 + n_3 + n_4 + n_5 - n_6 + n_7 \\
\lambda_5 &= n_1 - n_2 + n_3 - n_4 - n_5 + n_6 - n_7 \\
\lambda_6 &= n_1 + n_2 - n_3 + n_4 + n_5 + n_6 - n_7 \\
\lambda_7 &= -n_1 + n_2 + n_3 - n_4 + n_5 - n_6 - n_7 \\
\lambda_8 &= -n_1 - n_2 - n_3 + n_4 - n_5 - n_6 - n_7
\end{aligned} \tag{31}$$

The upshot of this analysis is that without loss of generality we can take the flux to be such that θ is given either by (28) in terms of four coefficients m_a, corresponding to

$$\xi = m_1 dx^{129} + m_2 dx^{349} + m_3 dx^{569} + m_4 dx^{789} \tag{32}$$

or by (30) in terms of seven coefficients n_a, corresponding to

$$\begin{aligned}
\xi = {} & n_1 dx^{123} + n_2 dx^{145} + n_3 dx^{167} + n_4 dx^{246} + \\
& + n_5 dx^{257} + n_6 dx^{347} + n_7 dx^{356}
\end{aligned} \tag{33}$$

Here $dx^{ijk} = dx^i \wedge dx^j \wedge dx^j$. It is now straightforward to analyse the supersymmetry of the solution given by the ansatz (32) or (33).

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It will be useful to define $\theta_{(i)}$ by

$$\theta\gamma_i = \gamma_i \theta_{(i)} \tag{34}$$

for each i, so that

$$9\Theta^2\Gamma_i + 6\Theta\Gamma_i\Theta + \Gamma_i\Theta^2 = \gamma_i \left(9\theta_{(i)}^2 + 6\theta_{(i)}\theta + \theta^2\right) \otimes 1\,. \tag{35}$$

If θ is chosen so that $\theta_{(i)}$ commutes with θ, as can be shown to be the case for the ansätze for θ (32) or (33), this can be rewritten as

$$\gamma_i U_{(i)}^2 \otimes 1\,, \qquad U_{(i)} \equiv 3\theta_{(i)} + \theta\,. \tag{36}$$

Then (19) implies

$$\left(-144\sum_j A_{ij}\gamma_j + \gamma_i U_{(i)}^2\right)\chi_- = 0\,. \tag{37}$$

for each $i = 1, \ldots, 9$, with, as seen above, no condition on χ_+. In a basis in which $U_{(i)}^2$ is diagonal for each $i = 1, \ldots, 9$, this can only have solutions if A_{ij} is also diagonal,

$$A_{ij} = -\operatorname{diag}\left(\mu_1^2, \mu_2^2, \ldots, \mu_9^2\right) \tag{38}$$

for some constants μ_i. Let the skew eigenvalues of $U_{(i)}$ be $\rho_{I(i)}$, $I = 1, \ldots, 8$, so that $U_{(i)}^2$ is a symmetric matrix with eigenvalues $-\rho_{I(i)}^2$, each with 2-fold degeneracy. Then if χ_- is chosen as an eigenvector χ_I satisfying

$$U_{(i)}^2\chi_I = -\rho_{I(i)}^2\chi_I \tag{39}$$

for some I, then it defines a Killing spinor providing that A_{ij} is given by (38) with the 9 coefficients μ_i determined to be

$$144\mu_i^2 = \rho_{I(i)}^2\,. \tag{40}$$

There will be (at least) 2 such extra Killing spinors, as each eigenvalue has (at least) two-fold degeneracy.

Given this choice of A_{ij}, a second pair of eigenspinors χ_J $(J \neq I)$ will also give extra Killing spinors if and only if

$$\rho_{J(i)}^2 = \rho_{I(i)}^2 \tag{41}$$

for each i. If there is an N-fold degeneracy in the eigenvalues $\rho_{I(i)}^2$,

$$144\mu_i^2 = \rho_{J_1(i)}^2 = \rho_{J_2(i)}^2 = \cdots = \rho_{J_N(i)}^2 \tag{42}$$

for all i, then there are $2N$ such extra Killing spinors, and the solution will have a total of $16 + 2N$ Killing spinors.

Given a flux defined by ξ, and any choice of I, the matrix A_{ij} can be chosen as in (38), (40) so that the two spinors χ_I with Θ^2 eigenvalues $-\lambda_I^2$ are Killing spinors. Next we turn to the conditions for degeneracy, (41). In a basis in which the anti-symmetric matrix θ is skew-diagonal with skew eigenvalues λ_I, then $\theta_{(i)}$ is also skew-diagonal for either ansatz (32) or (33); let its eigenvalues be $\lambda_{I(i)}$, and define $k_{I(i)}$ by

$$\lambda_I + \lambda_{I(i)} = 2k_{I(i)} \, . \tag{43}$$

Then

$$\rho_{I(i)} = 3\lambda_{I(i)} + \lambda_I = -2\left(\lambda_I - 3k_{I(i)}\right) \, . \tag{44}$$

For ansatz (32) $\rho_{I(9)} = 4\lambda_I$, while for ansatz (33), $\rho_{I(8)} = \rho_{I(9)} = -2\lambda_I$, so that (41) implies

$$\lambda_I^2 = \lambda_J^2 \tag{45}$$

and hence either $\lambda_I = \lambda_J$, or $\lambda_I = -\lambda_J$. If $\lambda_I = \lambda_J = 0$, then (41), (44) implies

$$k_{I(i)}^2 = k_{J(i)}^2 \tag{46}$$

for each i. If $\lambda_I = \lambda_J \neq 0$, then (41), (44) implies

$$\left(k_{I(i)} - k_{J(i)}\right)\left(3k_{I(i)} + 3k_{J(i)} - 2\lambda_I\right) = 0 \tag{47}$$

so that either $k_{I(i)} = k_{J(i)}$, or $3k_{I(i)} + 3k_{J(i)} = 2\lambda_I$. Similarly, if $\lambda_I = -\lambda_J \neq 0$, then either $k_{I(i)} = -k_{J(i)}$, or $3k_{I(i)} - 3k_{J(i)} = 2\lambda_I$.

Let us now analyse the 4-parameter ansatz (32) in detail, before briefly returning to the 7-parameter case (30) at the end. For (32), the eigenvalues λ_I (29) are of the form

$$\lambda_I = \sum_a L_{Ia} m_a \tag{48}$$

where each coefficient $L_{Ia} = \pm 1$. Then for $i = 1, \ldots, 8$,

$$k_{I(i)} = L_{Ia} m_a \, , \qquad \text{with} \qquad a = \left[\frac{i+1}{2}\right] \tag{49}$$

where a is the integer part of $(i+1)/2$, so that for example $k_{I(3)} = k_{I(4)} = L_{I2} m_2$. Then $k_{I(i)}^2 = m_a^2$ with $a = [(i+1)/2]$. This implies (46) for all I, J. Thus if $\lambda_I = \lambda_J = 0$, then χ_I and χ_J are both Killing spinors, provided the μ_i are chosen as in (40), and $N \geq 2$.

For at least one value of a, $L_{Ia} = L_{Ja}$, while for at least one other value of a $L_{Ia} = -L_{Ja}$, so that for some i, $k_{I(i)} = k_{J(i)}$ and for others $k_{I(i)} = -k_{J(i)}$. Consider next the case $\lambda_I = \lambda_J \neq 0$. Then for those i for which $k_{I(i)} = k_{J(i)}$, (41) is satisfied, while for those i for which $k_{I(i)} = -k_{J(i)}$, (47) implies that $k_{I(i)} = 0$, which in turn implies $m_a = 0$ for those $a = [(i+1)/2]$. Similarly, if $\lambda_I = -\lambda_J \neq 0$, then for those i such that $k_{I(i)} = k_{J(i)}$ the masses m_a must vanish for those $a = [(i+1)/2]$.

Thus if A_{ij} is chosen so that $\chi_- = \chi_I$ defines an extra Killing spinor, then the conditions (with the 4-parameter ansatz) that χ_J gives rise to a further Killing spinor are as follows. First, $\lambda_I^2 = \lambda_J^2$. If $\lambda_I = 0$, no further conditions are needed. If $\lambda_I \neq 0$, then one or more of the mass parameters m_a must be set to zero.

We will now summarise the degeneracy numbers N for each case, and so give the total number $16 + 2N$ of Killing spinors. First, for generic A_{ij}, there are no extra Killing spinors, $N = 0$, and there are just the 16 standard Killing spinors. If the A_{ij} are chosen as in (38), (40) for some I, then $N \geq 1$ and there are at least two extra Killing spinors, giving at least 18 supersymmetries. If one of the parameters is set to zero, $m_4 = 0$ say, then $N = 2$ and there are at least 20 Killing spinors. If two parameters are set to zero, $m_3 = m_4 = 0$ say, then $N = 4$ and there are 24 supersymmetries. If three masses are set to zero, then $N = 8$ and the maximally supersymmetric solution of [3] is recovered. If, on the other hand, the parameters are chosen to give $\lambda_I = 0$ but with all m_a non-zero, then there are no extra supersymmetries. If the masses are chosen so that in addition some of the other eigenvalues λ_J vanish, then there will be further supersymmetries. The parameters m_a can be chosen with them all non-zero so that 1, 2 or 3 of the eigenvalues λ_I vanish, giving $N = 1, 2$ or 3 and so 18, 20 or 22 supersymmetries respectively.

The Killing spinors take the form (15) where the spinor $\chi = (\chi_+, \chi_-)$ is given in terms of constant spinors ψ_+, ψ_- by (23). The spinors ψ_+ are unconstrained, giving 16 standard Killing spinors, while the spinors ψ_- are restricted to be $2N$ eigenspinors of $U^2_{(i)}$ with eigenvalue (40). Then on this $2N$ dimensional space, $U_{(i)} = 12\mu_i J$, where $J = \epsilon \otimes 1_{N\times N}$ and satisfies $J^2 = -1$. Then $\varepsilon = (\varepsilon_+, \varepsilon_-)$ with

$$\varepsilon_- = \chi_- \,, \qquad \varepsilon_+ = \chi_+ + \frac{1}{\sqrt{2}} \sum_i x^i \gamma_i \mu_i J \chi_- \tag{50}$$

Similarly, on this $2N$-dimensional space, θ is skew-diagonal with $\theta = \lambda J$ for some λ, so that

$$\chi_- = \exp\left(-\frac{1}{12} x^- \theta\right) \psi_- = \cos\left(\frac{\lambda x^-}{12}\right) \psi_- - \sin\left(\frac{\lambda x^-}{12}\right) J \psi_- \tag{51}$$

and ψ_- is restricted to lie in the $2N$-dimensional eigenspace. Note that if $\lambda = 0$ then the χ_- are independent of x^-. Similarly, ψ_+ can be decomposed into eigenspinors of θ, allowing the exponential in (23) to be calculated explicitly.

We now present explicit examples of the above cases. Let us first consider the cases with more than 18 supersymmetries obtained when one of the m_a is zero for generic non-vanishing λ_I. Without loss of generality we set $m_4 = 0$. For this case we have $\lambda_1 = -\lambda_2$ which corresponds to preservation of 20 supersymmetries in general. The solution is given by

$$\begin{aligned}
\xi &= m_1 dx^{129} + m_2 dx^{349} + m_3 dx^{569} \\
H &= -\frac{1}{36} (2m_1 - m_2 + m_3)^2 \left(x_1^2 + x_2^2\right) - \\
&\quad -\frac{1}{36} (-m_1 + 2m_2 + m_3)^2 \left(x_3^2 + x_4^2\right) - \\
&\quad -\frac{1}{36} (-m_1 - m_2 - 2m_3)^2 \left(x_5^2 + x_6^2\right) - \\
&\quad -\frac{1}{36} (-m_1 - m_2 + m_3)^2 \left(x_7^2 + x_8^2 + 4x_9^2\right)
\end{aligned} \tag{52}$$

Note that in general the extra Killing spinors will depend on all coordinates. If we set $m_1 = m_2 = -m_3$ then they are independent of $x_1, \ldots, x_6$. On the other hand if we set $m_3 = m_1 + m_2$ (which means $\lambda_1 = \lambda_2 = 0$) then they are independent of x_7, x_8, x_9 and x^-

and moreover four of the 16 Killing spinors (14) are then also independent of x^-. This has a metric which is the product of a pp-wave metric in eight dimensions with $\mathbb{E}^3$, but the flux has non-trivial components in one of the directions in $\mathbb{E}^3$, so that this solution can be regarded as a product of a nine-dimensional solution with $\mathbb{E}^2$.

If we further set $m_3 = 0$ then the solution preserves 24 supersymmetries in general and the extra Killing spinors will still depend on all coordinates. If $2m_1 = m_2$ then the extra Killing spinors are independent of x_1, x_2. On the other hand if $m_1 = -m_2$ (which means $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$) then they are independent of x_5, x_6, x_7, x_8, x_9 and x^-. In this case 8 of the 16 standard Killing spinors (14) are also independent of x^-. This has a metric which is the product of a pp-wave metric in six dimensions with $\mathbb{E}^5$, but again the flux has non-trivial components in one of the directions in $\mathbb{E}^5$. Finally, if we further set either $m_2 = 0$ or $m_1 = 0$ we recover the maximally supersymmetric solution of [3].

Next we consider cases of more than 18 supersymmetries with some $\lambda_I = 0$. In this case the extra Killing spinors defined by χ_I do not depend on x^9 or x^-. If we set $\lambda_1 = \lambda_2 = 0$ then we deduce $m_4 = 0$ which leads us back to one of the cases above with 20 supersymmetries. On the other hand if we set $\lambda_1 = \lambda_3 = 0$, which can be achieved by setting $m_1 = -m_2$ and $m_3 = m_4$, then we get something new. The solution now takes the form

$$\begin{aligned}
\xi &= m_1 \left(dx^{129} - dx^{349}\right) + m_3 \left(dx^{569} + dx^{789}\right) \\
H &= -\frac{1}{4}(m_1)^2 \left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right) - \\
&\quad -\frac{1}{4}(m_3)^2 \left(x_5^2 + x_6^2 + x_7^2 + x_8^2\right)
\end{aligned} \tag{53}$$

and generically preserves 20 supersymmetries. Note that for this case 4 of the 16 standard Killing spinors (14) are also independent of x^-.

An interesting special case is when in addition we have $\lambda_1 = \lambda_3 = \lambda_5 = 0$ which can be achieved by setting $m_1 = -m_2 = m_3 = m_4$. The solution then takes the form

$$\begin{aligned}
\xi &= m_1 \left(dx^{129} - dx^{349} + dx^{569} + dx^{789}\right) \\
H &= -\frac{1}{4}(m_1)^2 \left(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 + x_8^2\right)
\end{aligned} \tag{54}$$

and preserves 22 supersymmetries. For this case the extra 6 Killing spinors are independent of x^9 and x^- but are dependent on $x_1, \ldots, x_8$. In addition 6 of the 16 standard Killing spinors (14) are also independent of x^-. Note that if we reduce on the x_9 direction we obtain a type-IIA pp-wave solution that preserves 22 supersymmetries. This solution does not have a further Killing direction and so T-duality cannot be used to relate this to a IIB pp-wave solution. Thus such type-IIA pp-wave solutions with 22 supersymmetries could not be obtained by starting with IIB solutions and T-dualising as in [25]. Note that if we dimensionally reduce the D=11 solution along the x^- direction we obtain a type-IIA D0-brane solution that preserves 12 supersymmetries.

Finally we turn to the 7-parameter ansatz (30). For generic parameters, the A_{ij} can again be chosen to give 18 supersymmetries. Note that the expression for the Killing spinors (50), (51) discussed above for the 4-parameter case is also valid for this case. In the 7-parameter case, the analysis of the conditions for further supersymmetries is more complicated, but some simple cases can be analysed. If certain sets of four of the parameters vanish, e.g. if $n_4 = n_5 = n_6 = n_7 = 0$, then the ansatz is equivalent (after a relabelling) to the ansatz (29) with one of the parameters vanishing m_a, and hence gives at least 20 supersymmetries as discussed above. On the other hand, if certain sets of three parameters vanish e.g. $n_1 = n_2 = n_3 = 0$, then there are different configurations with 20 supersymmetries. The special case in which $n_4 = n_5 = -n_6 = n_7 = n$ leads to a solution

$$\begin{aligned} \xi &= n\left(dx^{246} + dx^{257} - dx^{347} + dx^{356}\right) \\ H &= -n^2\left(x_2^2 + x_3^2\right) \end{aligned} \tag{55}$$

In this case $\lambda_6 = \lambda_3 = 0$ and χ_6 and χ_3 are the extra Killing spinors, which are independent of x^-. Since $\lambda_1 = \lambda_2 = \lambda_7 = \lambda_8 = 0$ we conclude that 12 of the 16 standard Killing spinors are also independent of x^-. The extra Killing spinors are also independent of $x^4, \ldots, x^9$. This solution has a metric which is the product of a four dimensional pp-wave with $\mathbb{E}^7$ and the flux depends on three directions in $\mathbb{E}^7$.

It is interesting to observe that with the same ξ we can obtain a different solution preserving 20 supersymmetries:

$$\begin{aligned} \xi &= n\left(dx^{246} + dx^{257} - dx^{347} + dx^{356}\right) \\ H &= -\frac{1}{9}n^2\left(x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2\right) - \frac{4}{9}n^2\left(x_1^2 + x_8^2 + x_9^2\right) \end{aligned} \tag{56}$$

Now χ_4 and χ_5 are the extra Killing spinors and we have $\lambda_4 = -\lambda_5 = 4n$. This means that while 12 of the 16 standard Killing spinors are still independent of x^- the extra Killing spinors are not, and moreover depend on all coordinates.

Another case peserving 20 supersymmetries with $n_1 = n_2 = n_3 = 0$ can be obtained if we set $n_4 = n_5 = n_6 = n_7 = n$. The solution is given by

$$\begin{aligned} \xi &= n\left(dx^{246} + dx^{257} + dx^{347} + dx^{356}\right) \\ H &= -\frac{1}{9}n^2\left(x_1^2 + x_3^2 + x_5^2 + x_6^2 + x_8^2 + x_9^2\right) - \frac{4}{9}n^2\left(x_2^2 + x_4^2 + x_7^2\right) \end{aligned} \tag{57}$$

and since all λ_I are non-zero, all Killing spinors depend on x^-. The extra 4 Killing spinors depend on all x^i.

Finally we note that a solution with 22 supersymmetries can be found by setting all seven of the n_i to be equal to n. In this case the solution is given by

$$\begin{aligned} \xi &= n\left(dx^{123} + dx^{145} + dx^{167} + dx^{246} + dx^{257} + dx^{347} + dx^{356}\right) \\ H &= -n^2\left(x_2^2 + x_4^2 + x_6^2\right) - \frac{1}{4}n^2\left(x_8^2 + x_9^2\right). \end{aligned} \tag{58}$$

All λ_I are non-zero and thus all Killing spinors depend on x^-. The 6 extra Killing spinors do not depend on the co-ordinates x_1, x_3, x_5, x_7.

In conclusion we have demonstrated that there are solutions of M-theory with extra supersymmetries i.e. more than 16 and less than 32. In particular we have explicitly

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demonstrated solutions with 18, 20, 22 and 24 Killing spinors. It is possible that the seven-parameter ansatz (30) allows for further possibilities, but this seems unlikely. It is straightforward to see that the Penrose limits of various intersecting branes with $AdS \times$ Sphere factors explicitly discussed in [11] lead to special cases of our solutions. It would be interesting to know whether all of our solutions can be obtained as Penrose limits.

Now that the forbidden region of solutions preserving between 1/2 and all supersymmetries has been broached here and in [21, 25] it is natural to wonder, as in [28], whether all fractions are in fact obtainable. Perhaps the kind of analysis of [29] might provide some further insight into exotic fractions of supersymmetry.

Note added. In the final stages of writing up this work we became aware of [30] which has significant overlap with the work here.

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